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07. October 2019

Algorithms & Data Structures — Exercise sheet 3 — HS 19

Exercise Class (Room & TA): ______________________

Submitted by: ______________________

Peer Feedback by: ______________________

Points: ____

The solutions for this sheet are submitted at the beginning of the exercise class on October 14th.

Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

Exercise 3.1 *Counting Operations in Loops* **(2 Points)**.

For the following code fragments count how many times the function f is called. Report the number of calls as nested sum, and then simplify your expression in $\mathcal{O}$-notation (as tight and simplified as possible) and prove your result. For example, in the code fragment

Algorithm 1

for $k = 1, \dots, 100$ **do**
    $f()$

the function f is called $\sum_{k=1}^{100} 1 = 100$ times, so the amount of calls is in $\mathcal{O}(1)$.

a) Consider the snippet:

Algorithm 2

for $j = 1, \dots, n$ **do**
    for $k = j, \dots, n$ **do**
        $f()$

b) Consider the snippet:

Algorithm 3

for $j = 1, \dots, n$ **do**
    for $k = j, \dots, n$ **do**
        for $l = 1, \dots, 100$ **do**
            $f()$
            $f()$

c) Consider the snippet:

Algorithm 4

```
for k = 1, ..., 100 do
    f()
for j = 1, ..., n do
    f()
    for k = 1, ..., j do
        for l = 1, ..., j do
            for m = 1, ..., j do
                f()
```

d) Consider the snippet:

Algorithm 5

```
for j = 1, ..., n do
    for k = 1, ..., j do
        l ← 1
        while l ≤ j do
            f()
            l ← 2l
```

*e) Consider the snippet:

Algorithm 6

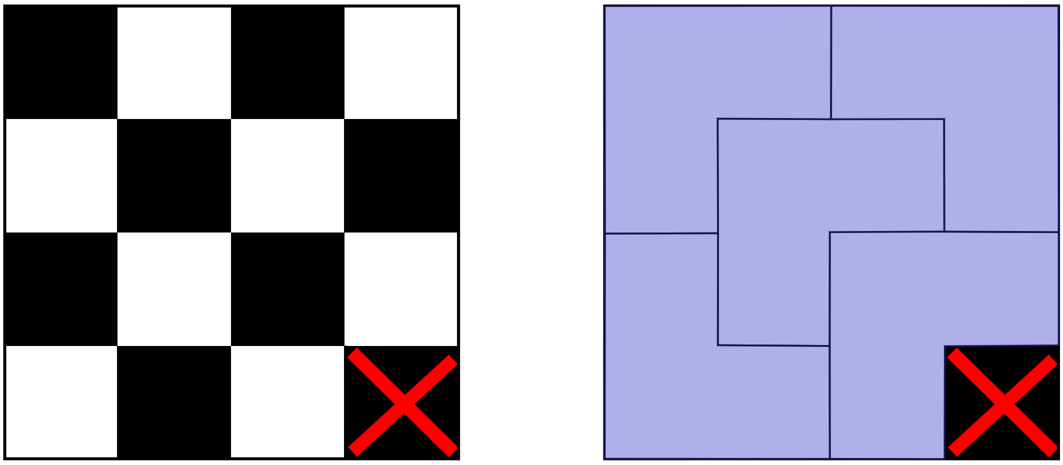
```
for j = 1, ..., n do
    for k = 1, ..., j do
        for ℓ = 1, ..., k do
            for m = ℓ, ..., n do
                f()
```

Exercise 3.2 *Divide and Conquer* **(1 Point)**.

a) List at least two algorithms from your solutions or the sample solutions of sheet 1 and sheet 2 that are divide-and-conquer algorithms.

b) Consider the following problem:

You are given a $2^k \times 2^k$ chessboard with one missing square and as many L-shaped puzzle pieces as you want. Each puzzle-piece can cover exactly three squares of the chessboard. As you will show algorithmically in this exercise, it is always possible to cover such chessboards by L-shaped puzzle pieces. An example is given in Figure 1 for $k = 2$, where the missing piece is a corner piece.



(a) 4×4 chessboard (b) covering

Figure 1: Example of a chessboard and its covering by L-shaped puzzle pieces.

1) Devise a divide-and-conquer algorithm that can cover a $2^k \times 2^k$ chessboard with one missing square at an arbitrary position for $k \in \{1, 2, 3, \dots\}$. Describe your algorithm using words. Make sure to describe how you divide the problem into *subproblems* and how you handle the *base case(s)*. Your description should be *concise* (e.g., it could have a pseudo-code-like form for readability).

   You can assume that each square is represented by its coordinates, specifically, the square in the lower left corner has coordinates $(1, 1)$ and the square in the upper right corner has coordinates $(2^k, 2^k)$. The input of your algorithm is (k, a, b), where a and b are coordinates of the missing square.

2) Determine the running time of your algorithm in terms of $n = 2^k$ in $\mathcal{O}$-notation.

Exercise 3.3* *Maximum-Submatrix-Sum.*

Provide an $\mathcal{O}(n^3)$ time algorithm which given a matrix $M \in \mathbb{Z}^{n \times n}$ outputs its maximal submatrix sum S. That is, if M has some non-negative entries,

$$S = \max_{\substack{1 \le a \le b \le n \\ 1 \le c \le d \le n}} \sum_{i=a}^{b} \sum_{j=c}^{d} M_{ij},$$

and if all entries of M are negative, $S = 0$.

Justify your answer, i.e. prove that the asymptotic runtime of your algorithm is $\mathcal{O}(n^3)$.

Hint: *You may want to start by considering the cumulative column sums*

$$C_{ij} = \sum_{k=1}^{i} M_{kj}.$$

How can you compute all C_{ij} efficiently? After you have computed C_{ij}, how you can use this to find S?